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Non-linear Vacuum Phenomena in Non-commutative QED

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Abstract

We show that the classic results of Schwinger on the exact propagation of particles in the background of constant field-strengths and plane waves can be readily extended to the case of nocommutative QED. It is shown that non-perturbative effects on constant backgrounds are the same as their commutative counterparts, provided the on-shell gauge invariant dynamics is referred to a non-perturbatively related space-time frame.

For the case of the plane wave background, we find evidence of the effective extended nature of non-commutative particles, producing retarded and advanced effects in scattering. Besides the known 'dipolar' character of non-commutative neutral particles, we find that charged particles are also effectively extended, but they behave instead as 'half-dipoles'.

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Introduction and Conclusions

In a classic paper [1], J. Schwinger solved, to leading order in $\hbar$ and to all orders in the coupling constant e, the QED non-linear effects on particle propagation in a couple of particular background configurations: constant field strengths, and plane waves. Using the proper time method, the classical non-linearities can be exactly summed at tree level, defining exact Green's functions for charged particles in these backgrounds. One can obtain from these 'dressed propagators' various important physical quantities such as dispersion relations, the Euler–Heisenberg effective Lagrangian to all orders, and the decay of a coherent electric field by pair production.

This paper is devoted to a generalization of Schwinger's results to the same background configurations in non-commutative QED. Although we mostly consider tree-level effects, our results are non-perturbative in the coupling constant, thus providing some non-perturbative information on the dynamics of the non-commutative theory.

At a purely perturbative level, non-commutative field theories (NCFTs) are defined by simply replacing standard products of fields by their Moyal products [2]:

$$\psi(x) \star \varphi(x) = \exp\left(\frac{i}{2}\theta^{\alpha\beta}\frac{\partial}{\partial\xi^\alpha}\frac{\partial}{\partial\eta^\beta}\right)\psi(x+\xi)\,\varphi(x+\eta)\Big|_{\xi=\eta=0}. \tag{1}$$

This introduces a peculiar non-local structure on length scales of $O(\sqrt{\theta})$ due to the non-commutativity of the coordinates [3]

$$[x^\alpha, x^\beta] = i\,\theta^{\alpha\beta}. \tag{2}$$

From the physical point of view, the most important effect of the non-local Moyal product is to give particles an effective extension, depending on the amount of energy-momentum they carry in the conjugate directions, in the sense of eq. (2). This leads to a topological classification of Feynman diagrams in NCFTs, [4] and to many other properties that are reminiscent of open-string dynamics. Indeed, one of the main interests of these theories is as toy models of string dynamics, without the complications of the gravitational sector. The precise relation between NCFTs and string theory involves backgrounds with large Neveu–Schwarz B-field fluxes, a much-studied subject recently [5, 6, 7, 8, 11].

One of the unexpected surprises of the perturbative studies was the lack of decoupling of non-commutative effects in generic theories [9, 10], at least within perturbation theory. In particular, the precise way in which the NCFTs negotiate their infrared singularities at a non-perturbative level is an important open question. Another interesting surprise was the discovery of inconsistencies in NCFTs when time is involved in the non-commutativity [11, 13], unless the effects of (2) are masked by stringy fuzziness. For instance, a concrete problem is the lack of unitarity of the S-matrix at a perturbative level [12]. There is also evidence that the same problems persist at the level of the large-N master field [13].

In this context, it is thus interesting to study some non-perturbative effects in a characteristic NCFT, such as the non-commutative version of QED. One of the distinctive non-perturbative effects described by Schwinger, the decay of a constant electric field via

pair production, has a string analog with special features, namely the decay rate diverges at a critical value of the electric field [14]. This is the same critical field that governs the classical singularity of the Born–Infeld effective action [15]. The relation between electric fields in open-string theory and time/space non-commutativity makes this example even more interesting. One of our chief objectives will be the search for similar features in NCFT.

The propagation of charge-q particles in the background of a gauge field $A(x)$ is controlled by the spectral properties of the covariant-derivative operators $D_{A\star}^{(q)}$ with

$$
\begin{aligned}
D_{A(x)\star}^{(1)} &= \partial_x + ieA(x)\star, \\
D_{A(x)\star}^{(0)} &= \partial_x + ie[\star A(x)\star,]
\end{aligned}
\tag{3}
$$

for the cases of neutral $q = 0$ matter, photons and ghosts, and the usual charged $q = 1$ matter, such as Dirac fermions or charged scalars. In particular, we are interested in the effective world-line Hamiltonian

$$
H(D_{A\star}^{(q)}) = -(D_{A\star}^{(q)})^2 + (V_{\text{eff}}^{(q)})\star,
\tag{4}
$$

with $V_{\text{eff}}^{(q)}$ a spin- and charge-dependent effective potential, linear in the non-commutative field strength $\widehat{F}$:

$$
\widehat{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie\, A_\mu \star A_\nu - ie\, A_\nu \star A_\mu.
\tag{5}
$$

For scalar and vector fields, H is the kinetic kernel of fluctuations in the $A(x)$ background. For spin $\frac{1}{2}$ particles, the effective world-line Hamiltonian $H(\not{D}_{A\star})$ is given by

$$
H(\not{D}_{A\star}) = (i\not{D}_{A\star})^2 - m^2 = -(D_{A\star})^2 - m^2 - \frac{e}{2}\,\sigma_{\mu\nu}\widehat{F}^{\mu\nu}\star,
\tag{6}
$$

and is related to the Dirac operator by the non-commutative generalization of the usual relation

$$
\frac{1}{i\not{D}_{A\star} - m} = \frac{i\not{D}_{A\star} + m}{(i\not{D}_{A\star})^2 - m^2} = (i\not{D}_{A\star} + m)\frac{1}{H(\not{D}_{A\star})}.
\tag{7}
$$

We wish to study the Green's function for $H(D_{A\star})$,

$$
G_{A\star}(x, y) = \langle x|\frac{1}{H(D_{A\star})}|y\rangle,
\tag{8}
$$

or 'dressed' propagator to all orders in the coupling constant (a different notion of non-commutative Green's function was developed in [16]). We shall also discuss the one-loop 'effective action' (in practice, various subtractions are needed in its definition):

$$
\Gamma[A\star] = (-1)^{2J+1}\frac{1}{2}\operatorname{Tr}\log H(D_{A\star}),
\tag{9}
$$

for each field of spin J in the matter and gauge sectors, including the ghosts. In this equation the trace 'Tr' is over the space-time, spin and internal quantum numbers.

Both the dressed Green's function and the effective action can be obtained from the heat kernel $K_{A\star}(x, y; s)$, defined by the equations

$$\left(\frac{d}{ds} + H(D_{A\star})\right) K_{A\star}(x, y; s) = 0, \qquad \lim_{s \to 0} K_{A\star}(x, y; s) = \delta(x - y), \qquad (10)$$

using the standard formal expressions:

$$G_{A\star}(x, y) = \int_0^\infty ds \, \langle x | e^{-sH(D_{A\star})} | y \rangle = \int_0^\infty ds \, K_{A\star}(x, y; s),$$

$$\text{Tr} \log H(D_{A\star}) = \int dx \int_0^\infty \frac{ds}{s} \, \text{tr} \, \langle x | e^{-sH(D_{A\star})} | x \rangle = \int_0^\infty \frac{ds}{s} \int dx \, \text{tr} \, K_{A\star}(x, x; s), \quad (11)$$

where 'tr' is now a trace over internal and spin quantum numbers.

At a technical level, the non-commutative problems (8-11) can be mapped to commutative problems with appropriate effective background configurations that are still exactly solvable. From the physical point of view, we find rather similar physical effects, depending on the non-commutative deformation parameter θ only through the non-commutative field strength $\widehat{F}_{\mu\nu}$, provided the space-time quantum numbers are properly interpreted.

It is important to stress that we are not making use of the mapping to commutative variables of ref. [7]. Our mapping to commutative variables is tailored to the problem of determining the dynamics of a probe particle in our restricted class of backgrounds.

In the case of constant field-strength background, the identification of good gauge invariant quantum numbers is non-trivial. This is due to the known fact that non-commutative gauge theories have no strictly local gauge-invariant operators. Although covariant local expressions are easy to devise, the construction of the gauge-invariant trace involves an integration over spacetime, neglecting surface terms at infinity, a problematic procedure for constant fields, since they do not turn-off at infinity.

We find that asymptotic on-shell quantities, such as dispersion relations, can be defined unambiguously, as well as integrated quantities over all spacetime, such as the real and imaginary parts of the effective *action*. This is again reminiscent of string theory, where the introduction of local sources is notoriously difficult. The proper definition of these gauge invariant quantities involves the use of a certain non-perturbative geometrical frame, different from the perturbative one. This is one of our main results; the appropriate notion of 'energy', as well as the causality structure, depends on the background field strength and the non-commutative deformation parameter in a precise way.

In the case of the plane-wave background we find that physical properties are characterized by the extended nature of probe particles; the known 'dipole' structure of non-commutative quantum-field quanta [17]. We also find a novel phenomenon for charged particles; namely there is a similar extensivity effect, with the particles behaving as 'half-dipoles'. The one-loop effective action in the plane-wave background is found to be trivial up to boundary terms. In the non-commutative case, this result depends on a judicious choice of ultraviolet and infrared cutoffs, reflecting the UV/IR connection pointed out in [9, 10].

Constant Field-Strength Background

Consider the family of field configurations of the form

$$A_\mu(x) = -\frac{1}{2} F_{\mu\alpha} x^\alpha = -\frac{1}{2} (Fx)_\mu, \tag{12}$$

with $F_{\mu\nu}$ *not necessarily* antisymmetric. The corresponding non-commutative field strength (5) is given by

$$\widehat{F}_{\mu\nu} = \left(F_A - \frac{e}{4} F\theta F^t \right)_{\mu\nu}, \tag{13}$$

where F_A denotes the antisymmetric part of F and F^t is the transpose. For example, if we consider a fully aligned magnetic 2×2 block, such that F has a magnetic flux on the same plane as $\theta^{\mu\nu}$ or, in other words $\mathbf{B} \parallel \boldsymbol{\theta}_m$, we have, restricting to that plane:

$$F = \begin{pmatrix} S_1 & B \\ -B & S_2 \end{pmatrix}, \qquad \theta = \begin{pmatrix} 0 & \theta_m \\ -\theta_m & 0 \end{pmatrix}, \qquad \widehat{F} = \begin{pmatrix} 0 & \widehat{B} \\ -\widehat{B} & 0 \end{pmatrix}, \tag{14}$$

then the non-commutative magnetic field $\widehat{B}$ is related to the parameters in A_μ and θ by

$$\widehat{B} = B - \frac{e}{4} \theta_m \left(B^2 + S_1 S_2 \right). \tag{15}$$

We see that, for $\theta \neq 0$, the symmetric part of F does contribute non-trivially to $\widehat{F}$, unlike the commutative case. In fact, including the symmetric part of F in the gauge potential is essential in obtaining non-commutative magnetic fields of arbitrary magnitude and sign.

An important property of (13) is its quadratic nature, so that different gauge potentials can lead to the same $\widehat{F}$, *i.e.* we can 'complete the square' and write

$$e\,\widehat{F} + \frac{1}{\theta} = \frac{1}{\theta} \left(1 + \frac{e}{2}\theta\,F \right) \left(1 - \frac{e}{2}\theta\,F^t \right) = \left(1 + \frac{e}{2}F\,\theta \right) \left(1 - \frac{e}{2}F^t\,\theta \right) \frac{1}{\theta}. \tag{16}$$

For instance, if $F = F_A$ is antisymmetric, we have two solutions (in fact more) corresponding to F and F', related by:

$$1 + \frac{e}{2}F\,\theta = -1 - \frac{e}{2}F'\,\theta. \tag{17}$$

In particular, a non-zero antisymmetric F satisfying $1 + \frac{e}{2}F\,\theta = 0$ corresponds to a vanishing non-commutative field strength $\widehat{F} = 0$. Therefore, it is natural to expect that these configurations are pure gauge. In this respect, we point out that the antisymmetric part of F is not the commutative field strength F_{SW} of the Seiberg–Witten mapping [7]. The precise relation is

$$e\,\widehat{F} + \frac{1}{\theta} = \frac{1}{\theta} \left(1 + \frac{e}{2}\theta\,F \right) \left(1 - \frac{e}{2}\theta\,F^t \right) = \frac{1}{\theta \left(\frac{1}{\theta} - e\,F_{\mathrm{SW}} \right) \theta}. \tag{18}$$

Notice that the mapping $\widehat{F}(F_{\mathrm{SW}})$ is one-to-one if we interpret the previous equation for arbitrary field strengths in the sense of analytic continuation. Thus, from the point of view of the Seiberg–Witten mapping, $\widehat{F} = 0$ is equivalent to $F_{\mathrm{SW}} = 0$, when written in commutative variables, and should characterize the vacuum unambiguously.

Gauge Ambiguity of the Linear Ansatz

The existence of discrete symmetries of eq. (5) that look like gauge transformations motivates a systematic discussion of the gauge ambiguity of the linear ansatz (12). Appart from the trivial gauge freedom of adding a constant to the gauge potential, let us find out the conditions for two gauge potentials of the form

$$A = -\frac{1}{2} F x, \qquad A^g = -\frac{1}{2} F^g x, \tag{19}$$

to actually be gauge-equivalent, namely the existence of a star-unitary transformation $g(x)$ satisfying

$$A^g = g^{-1} \star A \star g - \frac{i}{e} g^{-1} \star \partial g, \tag{20}$$

or, star-multiplying by $g(x)$ on the left:

$$ie\, g \star A^g = (\partial + ie\, A\star)\, g.$$

Notice that, at this point, we are not assuming any particular form for F and F^g; they are not antisymmetric in general.

Plugging in the linear ansatz (19) and using the formulas for left-right star products:

$$
\begin{aligned}
x^\mu \star g(x) &= \left(x^\mu + \frac{i}{2}\theta^{\mu\nu}\partial_\nu \right) g(x), \\
g(x) \star x^\mu &= \left(x^\mu - \frac{i}{2}\theta^{\mu\nu}\partial_\nu \right) g(x),
\end{aligned}
\tag{21}
$$

we arrive at

$$\left(1 + \frac{e}{4}(F + F^g)\,\theta \right) \partial g = -i\frac{e}{2}\,(F - F^g)\,x\,g. \tag{22}$$

Assuming that the matrix in the left hand side is invertible, we can write this equation in the form

$$(\partial + ie\, A_{\text{eff}})\, g = 0, \tag{23}$$

with

$$A_{\text{eff}}(x) = -\tfrac{1}{2}\, F_{\text{eff}}\, x \equiv -\tfrac{1}{2}\, \frac{1}{1 + \frac{e}{4}(F + F^g)\theta}\, (F^g - F)\, x. \tag{24}$$

Namely, the gauge transformation is covariantly constant (in the ordinary sense) with respect to a linear potential given by (24). The function $g(x)$ is solved as a Wilson line in A_{eff} but, in order to really define a gauge-transformation function, it should be independent of the integration path. This only happens if the associated field-strength *vanishes*. In other words, we need

$$\left(\frac{1}{1 + \frac{e}{4}(F + F^g)\theta}\, (F^g - F) \right)_{\text{antisym}} = 0. \tag{25}$$

Assuming, again, that the matrix in the denominator is non-singular, and noting the antisymmetry θ, this condition is equivalent to

$$F_A - \frac{e}{4} F \,\theta\, F^t = (F^g)_A - \frac{e}{4} F^g \,\theta\, (F^g)^t.$$

Thus, we obtain the expected result that two linear potentials are gauge-equivalent when they have the same field strengths: $\widehat{F}(F) = \widehat{F}(F^g)$. The associated gauge transformation may be written as

$$g(x) = \exp\left(\frac{ie}{4} x \left(F_{\text{eff}}\right) x + \dots\right), \tag{26}$$

where the dots represent a trivial constant and a possible linear term, that simply adds a constant to the gauge potentials. Restricting now to antisymmetric F, we seem to have found that F and F' related by (17) are gauge equivalent. However, the $\mathbf{Z}_2$ transformation $F \to F'$ is *singular* as a gauge transformation since, precisely for $F^g = F'$, one has

$$1 + \frac{e}{4}(F + F')\,\theta = 0,$$

and the effective connection A_{eff} in (24) is not defined. Thus, if we restrict ourselves to *non-singular* gauge transformations, we would be forced to conclude that the $\mathbf{Z}_2$ transformation is not gauge and both branches are physically different. According to this interpretation of our formulae, there would be a physical 'screening effect' consisting on the neutralization of $\widehat{F}$ by the θ-background, a conclusion that would be at odds with our previous considerations based on the Seiberg–Witten mapping. On the other hand, it is easy to see that the $\mathbf{Z}_2$ transformation (17) is a member of a continuous set of regular gauge transformations that do not change $\widehat{F}$. To be more specific, consider the magnetic aligned ansatz of eq. (14), with the restriction $S_1 = S_2 = S$ for simplicity. Then, the potentials that are gauge-equivalent to the vacuum $\widehat{B} = 0$ lie on a circle of radius $2/e\theta_m$ in the (B, S) plane, centered at the point $(B = 2/e\theta_m, S = 0)$. There is a gauge transformation of the form (26) that takes the vacuum at $S = B = 0$ into any of the points lying on the $\widehat{B} = 0$ circle. Of these, the gauge transformation that takes $B = S = 0$ to $S^g = 0, B^g = 4/e\theta_m$ is singular. It is also the only singular one, since the determinant

$$\det\left(1 + \frac{e}{4} F^g\,\theta\right) = \left(B^g - \frac{4}{e\theta_m}\right)^2 + (S^g)^2 = 0$$

only vanishes for $S^g = 0, B^g = 4/e\theta_m$. Namely, the singular gauge transformation is a member of a continuous *regular* family of gauge transformations. Since it can be approximated by honest gauge transformations and it does not introduce singularities in physical, gauge-invariant quantities, it should be allowed. It should be interesting to study this question from the point of view of the Wilson-loop operators defined in [18].

Reduction to the Commutative Frame

The basic property of the linear ansatz (12), allowing us to solve the problem for constant fields, is the general form of the Moyal product by a single variable:

$$x^\mu \star = x^\mu + \frac{i}{2}\theta^{\mu\nu}\partial_\nu \qquad [x^\mu \star, \ ] = i\theta^{\mu\nu}\partial_\nu. \tag{27}$$

Using these relations we can map the basic covariant derivatives (3) into ordinary differential operators. It is convenient to distinguish the cases of charged, $q = 1$, and neutral, $q = 0$, probe particles.

Charge One

Direct application of the first equation in (27) leads to

$$\partial_x + ieA(x)\star = \partial_x - \frac{ie}{2}F\,x\star = \left(1 + \frac{e}{4}F\theta\right)\partial_x - \frac{ie}{2}F\,x.$$

It is convenient to define the new coordinates x'^μ by the relation

$$x^\nu = x'^\mu \left(1 + \frac{e}{4}F\,\theta\right)^\nu_{\ \mu}. \tag{28}$$

Using the antisymmetry of θ, we can write the transformation matrix in appropriate form for left-action on contravariant vectors, $x = M\,x'$ with

$$\left(M_{(q=1)}\right)^\mu_{\ \nu} = \left(1 - \frac{e}{4}\theta\,F^t\right)^\mu_{\ \nu}. \tag{29}$$

Finally, we apply the identity

$$\widehat{F} = F_A - \frac{e}{4}F\,\theta\,F^t = F\,M_1 - F_S, \tag{30}$$

with F_S the symmetric part of F, obtaining the following reduction formula to a purely commutative operator:

$$D^{(1)}_{A(x)\star} \equiv \partial_x + ieA(x)\star = \partial_{M_1^{-1}x} + ie\widehat{A}(M_1^{-1}x) \equiv D_{\widehat{A}(x')}, \tag{31}$$

with the equivalent commutative gauge potential given by:

$$\widehat{A}(x') = -\frac{1}{2}\left(\widehat{F} + F_S\right)x'. \tag{32}$$

Thus, up to the expected replacement $F \to \widehat{F}$, we have just a linear transformation of the coordinates. Notice that, once we have a commutative operator, the symmetric part F_S in the previous formula is a pure $U(1)$ gauge ambiguity. We stress that the reduction formula is only valid provided M^{-1} exists.

The full non-commutative gauge symmetry (20), acting through Moyal products:

$$D_{A\star} \to D_{A^g\star} = g^{-1} \star D_{A\star} \star g, \tag{33}$$

is mapped after the reduction to an ordinary $U(1)$ action on $\widehat{A}$, plus a coordinate transformation on the x'^μ. Indeed, the transformation matrix M_1 is not gauge invariant, as

one can easily check for example using the aligned magnetic ansatz of (14) (we set $e = 4$ to simplify the notation)

$$M_1 = 1 - \theta \, F^t = \begin{pmatrix} 1 - \theta_m \, B & -\theta_m \, S_2 \\ \theta_m \, S_1 & 1 - \theta_m \, B \end{pmatrix}. \tag{34}$$

Not even the determinant

$$\det(M_1) = (1 - \theta_m \, B)^2 + \theta_m^2 \, S_1 S_2 = \frac{1}{2} - \theta_m \, \widehat{B} - \theta_m \left(B - \frac{1}{2\theta_m} \right)$$

is gauge invariant. Notice that M_1 is singular at the locus of $\det(M_1) = 0$, or

$$\det \left(1 - \frac{e}{4} \theta \, F^t \right) = \det \left(1 + \frac{e}{4} F \, \theta \right) = 0. \tag{35}$$

Hence, the singularities of the transformation to the commutative frame coincide (for the $q = 1$ case) with the singular gauge transforms of the vacuum.

Charge Zero

The neutral covariant derivative admits a similar reduction with the commutative covariant derivative being trivial

$$D^{(0)}_{A(x)\star} \equiv \partial_x + ie[\star A(x)\star, \;] = \partial_{M_0^{-1} x}, \tag{36}$$

and the transformation matrix differing by a rescaling of the coupling:

$$M_{(q=0)} = 1 - \frac{e}{2} \theta \, F^t. \tag{37}$$

In this case the transformation matrix M_0 is almost gauge invariant. Namely, one can rewrite (16) in the form

$$1 + e \, \theta \, \widehat{F} = \theta \, (M_0)^t \, \theta^{-1} \, M_0 = (M_0)^t \, \theta^{-1} \, M_0 \, \theta = 1 + e \, \widehat{F} \, \theta,$$

so that,

$$(\det(M_0))^2 = \det \left(1 + e \, \theta \, \widehat{F} \right) \tag{38}$$

is manifestly gauge-invariant (the individual entries of M_0 are, however, not gauge-invariant). Thus, in this case the singular locus of the transformation $x \to x'$, satisfying

$$\det \left(1 - \frac{e}{2} \theta \, F^t \right) = 0 = \det \left(1 + e \, \theta \, \widehat{F} \right), \tag{39}$$

is gauge-invariant. It is interesting that the singularity coincides with that of the Seiberg–Witten mapping to commutative field strengths F_{SW}, given in eq. (18).

The Green's Functions

We can use these results to write a reduction formula for the dressed propagator. From the defining equation

$$H(D_{A\star})\, G_{A\star}(x,y) = \delta(x-y),\tag{40}$$

we get (at this point, it is not necessary to specify the charge q)

$$G_{A\star}(x,y) = \frac{1}{|\det(M)|} G_{\widehat{A}}(M^{-1}x, M^{-1}y),\tag{41}$$

where the determinant factor comes from the transformation of the delta function, and $G_{\widehat{A}}$ is the Green's function of the commutative problem for the field $\widehat{A}$. Taking Fourier transforms:

$$\widetilde{G}_{A\star}(p) = \int dx\, e^{ipx}\, G_{A\star}(x) = \int \frac{dx}{|\det(M)|}\, G_{\widehat{A}}(M^{-1}x)\, e^{ipx} = \int dx\, G_{\widehat{A}}(x)\, e^{ipMx},$$

we obtain the momentum-space reduction formula:

$$\widetilde{G}_{A\star}(p) = \widetilde{G}_{\widehat{A}}(pM).\tag{42}$$

Alternatively, we can obtain the same results by considering of eigenvalue problem

$$H(D_{A(x)\star})\, \psi_n(x) = \lambda_n\, \psi_n(x).\tag{43}$$

Our reduction algorithm gives

$$H(D_{\widehat{A}(x')})\, \psi_n(Mx') = \lambda_n\, \psi_n(Mx'),\tag{44}$$

with $D_{\widehat{A}(x')}$ the corresponding commutative operator in the potential $\widehat{A}$ defined by eq. (32). Thus, the original and reduced operators actually have the same spectrum of eigenvalues. Denoting by $\widehat{\psi}_n(x)$ the normalized eigenfunctions of $H(D_{\widehat{A}})$, we obtain

$$\psi_n(x) = |\det(M)|^{-\frac{1}{2}}\, \widehat{\psi}_n(M^{-1}x),\tag{45}$$

where the precise proportionality constant comes from the unit normalization of the eigenfunctions. From here we can write the Green's function using its formal spectral definition:

$$\begin{aligned}
G_{A\star}(x,y) &= \sum_n \frac{\psi_n(x)\psi_n(y)^*}{\lambda_n} = \sum_n \frac{1}{|\det(M)|}\frac{\widehat{\psi}_n(M^{-1}x)\widehat{\psi}_n(M^{-1}y)^*}{\lambda_n} \\
&= \frac{1}{|\det(M)|}\, G_{\widehat{A}}(M^{-1}x, M^{-1}y),
\end{aligned}\tag{46}$$

in agreement with (41).

A similar relation follows for the heat kernel:

$$K_{A\star}(x,y;s) = \frac{1}{|\det(M)|}\, K_{\widehat{A}}(M^{-1}x, M^{-1}y; s).\tag{47}$$

Explicit expressions for the Green's functions can be written by applying the reduction formula (41) to Schwinger's result in the case of charge-one particles [1] (see also [19], pag. 100). The singularities of $G_{A\star}(x, y)$ at non-perturbative 'large' values of the gauge field $A(x)$ coincide with $\det(M_1) = 0$ and should be interpreted as gauge artifacts, since we found that such gauge potentials are singular gauge transforms of the vacuum.

The Green's function for $q = 0$ neutral particles is the simplest possible; in momentum space:

$$\widetilde{G}_{q=0}(p) = \frac{i}{(pM_0)^2 - m^2 + i0} = \frac{i}{(p')^2 - m^2 + i0}. \tag{48}$$

It is interesting that the causal structure inherited from the perturbative rule $m^2 \to m^2 - i0$ is the standard Feynman prescription, in terms of the rotated momenta $p' = pM_0$. We shall provide some perspective on this observation in the next section.

Dispersion Relations

The Green's functions found above are instrumental in constructing the perturbative expansion in the background of a constant non-commutative field-strength $\widehat{F}$. However, non-linear tree-level effects on particle propagation can be extracted directly from the gauge-invariant information contained in the singularities of the Green's function, *i.e.* the dispersion relations, that are nothing but the diagonalized version of the Klein–Gordon equations:

$$H(D_{A\star})\varphi(x) = 0. \tag{49}$$

Thus, the analysis of dispersion relations can be carried out quite generally without explicit knowledge of the full form of the Green's function. Using the general relation

$$\widetilde{G}_{A\star}(p) = \widetilde{G}_{\widehat{A}}(pM), \tag{50}$$

and the dispersion relation of the commutative problem:

$$f_{\widehat{A}}(p) = 0, \tag{51}$$

as determined by the singular locus of the momentum-space Green's function $\widetilde{G}_A(p)$, or by the solution of Klein–Gordon-like equation:

$$H(D_{\widehat{A}(x)})\phi(x) = 0, \tag{52}$$

we can immediately write down the corresponding dispersion relation for the non-commutative counterpart. It is given by:

$$f_{\widehat{A}}(pM) = 0, \tag{53}$$

where M is either M_0 or M_1 depending on the case we consider. In fact, the correct form of the dispersion relation is

$$f_{\widehat{A}}(p') = 0, \tag{54}$$

with

$$p'_\mu = p_\nu M^\nu{}_\mu. \tag{55}$$

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The reason why (54) is the correct form is that the momenta p_μ, conjugate to the original non-commutative coordinates x^μ, are not good quantum numbers for non-zero $\widehat{F}$ and θ. The dispersion relation cannot contain explicitly the transformation matrix M, because it is not gauge invariant.

We can illustrate these ideas with a couple of examples. If we deal with ordinary charge-one particles in a magnetic field pointing in the z-direction, we know the dispersion relation is independent of the momenta in the (x, y)-plane p_x, p_y. There is an infinite degeneracy of energy levels, labeled by *one* momentum variable, plus a discrete energy spectrum of Landau levels. Namely the states $|p_x, p_y\rangle$ are substituted, on diagonalizing the Hamiltonian, by the states $|p_y, n\rangle$, with the energy depending only on the oscillator quantum number n. Our choice of p_y as the continuous momentum label in the (x, y)-plane is a gauge-dependent choice, as any other linear combination of p_x and p_y would be valid, but the spectrum is gauge-invariant. One obtains, for a charge-one particle,

$$E^2 = m^2 + p_z^2 + e|B|(2n + 1 + \alpha), \tag{56}$$

with α related to spin quantum numbers, taking $2J + 1$ values.

Therefore, the non-commutative counterpart according to (53) is

$$((pM_1)_0)^2 = m^2 + ((pM_1)_z)^2 + e|\widehat{B}|(2n + 1 + \alpha). \tag{57}$$

In gauge-invariant form:

$$(E')^2 = m^2 + (p'_z)^2 + e|\widehat{B}|(2n + 1 + \alpha). \tag{58}$$

This example illustrates how the background fields determine what are the good quantum numbers of on-shell asymptotic states. Even in the commutatative case, we learn that only one linear combination of p_x, p_y is a good quantum number for the problem. In the non-commutative case, the gauge-invariant dispersion relation is written in terms of E' and p'_z which are the good energy and momentum variables. In addition, we have the discrete label of Landau levels n, and an infinite degeneracy labeled by one linear combination of p'_x and p'_y.

An even simpler example is given by the dispersion relation for neutral particles. In this case the pole is determined by

$$(pM_0)_\mu \, \eta^{\mu\nu} \, (pM_0)_\nu = m^2.$$

We can summarize this by defining an effective metric G_{eff}:

$$(G_{\text{eff}})^{-1} = M_0 \, \eta^{-1} \, M_0^t = \left(1 - \frac{e}{2}\theta \, F^t\right) \eta^{-1} \left(1 + \frac{e}{2}F \, \theta\right), \tag{59}$$

which would give M_0 the interpretation of a *vierbein*. The dispersion relation reads:

$$(G_{\text{eff}})^{\mu\nu} p_\mu p_\nu = m^2 \tag{60}$$

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Considering for example a purely antisymmetric F, one can readily check that the effective metric degenerates in an analog of the Born–Infeld singularities of string theory in background electric fields. One can also detect superluminal propagation, in the sense of $dE/d|\mathbf{p}| > 1$, for specific values of the background fields. In any case, such effective metric is not gauge invariant. Considering the general magnetic-aligned ansatz (14) with $e = 4$ one finds for the 2×2 magnetic block of the effective inverse metric:

$$(G_{\text{eff}})^{-1} = -M_0(M_0)^t = -\left(\begin{array}{cc} (1 - 2\theta_m B)^2 + 4\theta_m^2 S_2^2 & 2\theta_m (1 - 2\theta_m B)(S_1 - S_2) \\ 2\theta_m (1 - 2\theta_m B)(S_1 - S_2) & (1 - 2\theta_m B)^2 + 4\theta_m^2 S_1^2 \end{array} \right).$$

Since the gauge-invariant combination of B, S_1 and S_2 is

$$|\det(M_0)| = 1 - 4\theta_m \widehat{B} = (1 - 2\theta_m B)^2 + 4\theta_m^2 S_1 S_2,$$

we see that the individual entries of the effective metric are not gauge invariant. Therefore, even in the neutral case, we are led to defining new momentum quantum numbers for the problem in a deformed space:

$$p' = pM. \tag{61}$$

Whenever the entries M^{0i} of the transformation matrix are non-vanishing, the frame-transformation advocated here mixes the energy and momentum running in the free propagators of the weak coupling expansion. This happens even in the case of pure magnetic fields, provided the θ^{0i} components linking the time direction with the magnetic-flux plane are non-vanishing. The resummation of all tree interactions with the background, to all orders in the electromagnetic coupling e, produces effective momentum variables with standard causal structure.

Effective Action and Particle Production

According to our previous considerations, the spectrum of the covariant-derivative operators does not change under the reduction operation, and we expect the one-loop determinant of these operators to be formally equal:

$$\det H(D_{A(x)\star}) = \prod_n \lambda_n = \det H(D_{\widehat{A}(M^{-1}x)}). \tag{62}$$

Alternatively, using the general transformation rule for the heat kernel (47) we can read-off the effective action from the general expression (11). One obtains

$$\begin{aligned} \Gamma[A\star] &= \frac{(-1)^{2J+1}}{2} \int_0^\infty \frac{ds}{s} \int dx\, K_{A\star}(x, x; s) \\ &= \frac{(-1)^{2J+1}}{2} \int_0^\infty \frac{ds}{s} \int dx\, |\det(M)|^{-1}\, K_{\widehat{A}}(M^{-1}x, M^{-1}x; s). \end{aligned} \tag{63}$$

Changing variables to the x'-frame yields

$$\Gamma[A\star] = \frac{1}{2}(-1)^{2J+1} \int_0^\infty \frac{ds}{s} \int dx'\, K_{\widehat{A}}(x', x'; s) = \Gamma[\widehat{A}], \tag{64}$$

which is precisely Schwinger's result for the field-strength $\widehat{F}$. Notice that there is a certain ambiguity in this manipulation. Had we decided to use translational invariance of the heat kernel in (47) (*i.e.* the fact that it only depends on the difference $x - y$) we would have obtained a different result for the effective Lagrangian, by a factor of $|\det(M)|^{-1}$. The presence of such a factor would be rather problematic, since it is not gauge invariant for $q = 1$ particles. This factor would multiply the probability density of pair production (the imaginary part of Γ), as well as the logarithmic divergence of the real part of Γ. In the first case it would lead to a non-gauge-invariant decay rate. In the second case we would learn that, upon expanding in a power series in the coupling e, the theory needs an infinite number of high-derivative counterterms of the form $(\theta\widehat{F})^n$ to subtract logarithmic divergences. This would be at odds with the diagrammatic analysis of ultraviolet divergences, where one finds that switching on θ always results in an improved ultraviolet behaviour, rather than the contrary.

The difficulties in defining a gauge-invariant *integrand*, or effective *Lagrangian*, as opposed to just defining an effective *action*, is another manifestation of the lack of standard local gauge-invariant operators in these theories. The ambiguity mentioned above is likely to be related to the difficulties in handling total derivatives for constant fields. In particular, if we consider an adiabatic situation with a slowly varying field-strength, it is clear that the heat kernel is not translationaly invariant any more.

On the other hand, once we write the answer in the x'-frame as in (64), since we are actually integrating over x', and they are dummy variables, we may as well replace them by the original coordinates x. Thus we find a gauge-invariant density of decay probability given by Schwinger's result [1, 19]:

$$w(x) = (-1)^{2J+1}\,(2J+1)\,\frac{\alpha_{\text{em}}|\widehat{E}|^2}{2\pi^2}\,\sum_{n=1}^{\infty}\frac{(-1)^{n(2J+1)}}{n^2}\,e^{-(n\pi m^2/|e\widehat{E}|)}, \qquad (65)$$

where J is the spin of the particle pairs that are being produced.

Plane-Wave Background

In this section we consider the problem of particle propagation in the background of a plane wave. For simplicity, we take it to be linearly polarized:

$$A^\mu(x) = \varepsilon^\mu\,A(\xi), \qquad (66)$$

with $\xi = n^\mu x_\mu$, $n^2 = \varepsilon \cdot n = 0$, $\varepsilon^2 = -1$. It is easy to see that this background satisfies the classical equations of motion of non-commutative QED.

We use the notation $(x^\mu) = (t, x, y^1, y^2) \equiv (t, x, \mathbf{y})$ and choose coordinates so that θ is skew-diagonal with non-trivial entries

$$[x, y^1] = i\theta_m, \qquad [t, y^2] = i\theta_e. \qquad (67)$$

Using the remaining $SO(1,1) \times SO(2)$ symmetries of the problem, we can let the wave propagate in the (x, t)-plane:

$$n = (1, 1, 0, 0), \qquad \varepsilon = (\varepsilon^0, \varepsilon^0, \boldsymbol{\varepsilon}), \qquad \boldsymbol{\varepsilon}^2 = 1. \qquad (68)$$

An important simplification is that the field strength equals the commutative one:

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} = f_{\mu\nu} A'(\xi) = (n_\mu \varepsilon_\nu - n_\nu \varepsilon_\mu) A'(\xi), \tag{69}$$

where the prime stands for derivative with respect to $\xi = t - x$. In the following we consider the particular case of $q = 1$ Dirac fermions and $q = 0$ Majorana fermions, for the purposes of definiteness, although the results are easily generalized to particles of other spin assignments.

The Equivalent Commutative Problem

The Dirac operators $i\slashed{D}_{A\star}^{(q)}$, when acting on functions of the form

$$\psi_{\mathbf{k}}(t, x)\, e^{i\mathbf{k}\mathbf{y}},$$

become the standard-product operator

$$i\slashed{\partial}_{(t,x)} + \boldsymbol{\gamma} \cdot \mathbf{k} - m - e\slashed{\mathcal{A}}_{\mathbf{k}}^{(q)}, \tag{70}$$

with a shifted effective potential

$$\mathcal{A}_{\mathbf{k}}^{(q=1)} = A(nx - \tfrac{1}{2}\, n\theta k) \tag{71}$$

for the charged case, and

$$\mathcal{A}_{\mathbf{k}}^{(q=0)} = A(nx - \tfrac{1}{2}\, n\theta k) - A(nx + \tfrac{1}{2}\, n\theta k) \tag{72}$$

for the neutral case, where $k = (0, 0, \mathbf{k})$. The shifts act only on the null projection of $\theta\mathbf{k}$ on the (t, x) plane. Therefore, it is appropriate to use a mixed coordinate-momentum representation and analyze the effective two-dimensional effective operator (70) at fixed values of $\mathbf{k}$.

In the following, we just use $\mathcal{A}_{\mathbf{k}}(\xi)$ as a unified notation, keeping in mind that it depends implicitly on q and $\theta^{\mu\nu}$. Notice that a carefully tuned periodic wave can render the shift effects irrelevant for a particular value of the transverse momentum $\mathbf{k}$.

Scattering and Advanced/Retarded Effects

The scattering of particles of charge $q = 0, 1$ by the plane wave can be solved exactly by simply generalizing old textbook results (see for instance [19]). We wish to solve the non-commutative Dirac wave equation

$$(i\slashed{D}_{A\star}^{(q)} - m)\, \psi(x) = 0, \tag{73}$$

with A_μ as in (66). We assume that $A(\xi)$ is a function with compact support. This implies that in the asymptotic past $\psi(x)$ can be chosen to be an incoming monochromatic plane wave. With the choice for n^μ (68) we know that $\mathbf{k} = (k^1, k^2)$ is conserved. Scattering is purely one-dimensional and it will be reduced to computing the phase shifts of the outgoing

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wave-function after the particle traverses the wave-front. We can look for solutions of (73) of the form

$$\psi(x) = e^{-ik\cdot x}\,\psi_k(t,x), \tag{74}$$

with $k^\mu = (k^t, k^x, \mathbf{k})$ and $k^2 = m^2$ as implied by the asymptotic conditions. After the reduction to the commutative operator (70) we have the ordinary equation:

$$\left[i\slashed{\partial} - e\,\slashed{\mathcal{A}}_{\mathbf{k}} - m\right] e^{-ik\cdot x}\,\psi_k(t,x) = 0, \tag{75}$$

where $\mathcal{A}_{\mathbf{k}}$ is given by (71) for $q = 1$ and (72) for $q = 0$. Note that, although k is a four-momentum, in $n\theta k$ only $\mathbf{k}$ appears for $n = (1,1,0,0)$.

When $q = 1$, if $n\theta k > 0$, the particle sees the pulse a time $\frac{1}{2}\,|n\theta k|$ before the commutative counterpart would, whereas for $n\theta k < 0$, it will interact with it later than the commutative counterpart by the same amount $\frac{1}{2}\,|n\theta k|$.

The $q = 0$ case is more curious. For commutative neutral particles there is absolutely no interaction with the wave front. In the non-commutative case however the particle sees a pulse $A(\xi - \frac{1}{2}\,n\theta k) - A(\xi + \frac{1}{2}\,n\theta k)$, containing always and advanced and a retarded component of opposite signs. Only when the transverse momentum $\mathbf{k}$ vanishes the interaction disappears.

This once again hints at the interpretation of particles in non-commutative field theory as extended one-dimensional objects with charges at the endpoints. In the $q = 0$ case we can think of them as dipoles whose size depends on the transverse momentum $\mathbf{k}$, whereas for $q = 1$ they seem to behave as 'half-dipoles' also with an effective size depending on $\mathbf{k}$ (perhaps in this case it should be more appropriate to think of a dipole but with one end at infinity).

The solution of (75) is straightforward [19] and we do not dwell on the details. Picking a solution of the free Dirac equation $u(k)$ satisfying

$$(\slashed{k} - m)\,u(k) = 0, \qquad \bar{u}\,u = 2m,$$

we can write the result as:

$$\psi_k(t,x) = \left(1 + \frac{ie\,\slashed{n}\,\slashed{\mathcal{A}}}{2k\cdot n}\,\mathcal{A}_{\mathbf{k}}(\xi)\right) e^{iI_k}\,u(k), \tag{76}$$

where

$$I_k = -k\cdot x - \int_{-\infty}^{n\cdot x} d\xi\,\left(e\frac{\mathcal{A}_{\mathbf{k}}(\xi)\,\varepsilon\cdot k}{n\cdot k} + e^2\frac{\mathcal{A}_{\mathbf{k}}^2(\xi)}{2n\cdot k}\right) \tag{77}$$

is the eikonal of the particle, if we think in terms of geometrical optics. The phase shift now depends on the value of the transverse momentum of the particle scattered, and it incorporates the advanced and/or retarded effects of the wave front $\mathcal{A}_{\mathbf{k}}^{(q)}$ due to the effective extended nature of the particles.

The Green's Function

For a more detailed description of the propagation in the background of the plane wave we need to compute the Green's function

$$S(x, x') = \langle x | \frac{1}{i\slashed{D}_{A\star} - m} | x' \rangle = \int \frac{d\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}(\mathbf{y} - \mathbf{y}')} \, S_{\mathbf{k}}(t, x; t', x'), \tag{78}$$

where the mixed-representation Green's function is given by

$$S_{\mathbf{k}}(t, x; t', x') = \langle t, x | \frac{1}{i\slashed{\partial}_{(t,x)} + \boldsymbol{\gamma} \cdot \mathbf{k} - m - e\slashed{A}_{\mathbf{k}}} | t', x' \rangle. \tag{79}$$

We shall compute it through the bosonic one:

$$S_{\mathbf{k}}(t, x; t', x') = \left(i\slashed{\partial}_{(t,x)} + \boldsymbol{\gamma} \cdot \mathbf{k} - e\slashed{A}_{\mathbf{k}} + m \right) G_{\mathbf{k}}(t, x; t', x'),$$

with

$$G_{\mathbf{k}}(t, x; t', x') = \langle t, x | \frac{1}{\left(i\slashed{\partial}_{(t,x)} + \boldsymbol{\gamma} \cdot \mathbf{k} - e\slashed{A}_{\mathbf{k}} \right)^2 - m^2} | t', x' \rangle.$$

Following Schwinger's treatment in the proper-time method:

$$G_{\mathbf{k}}(t, x; t', x') = -i \int_{-\infty}^{0} d\tau \, \langle t, x | e^{-i\tau(H + i0)} | t', x' \rangle \tag{80}$$

where the effective world-line Hamiltonian is given by

$$H = \pi_a \pi^a + V, \tag{81}$$

with the definitions

$$\begin{aligned}
\pi_a &= i\partial_a - e\mathcal{A}_a = i\partial_a - e\varepsilon_a \mathcal{A}_{\mathbf{k}}(\xi), \\
V &= -\mathbf{k}^2 - m^2 + 2e\mathbf{k} \cdot \boldsymbol{\varepsilon} \, \mathcal{A}_{\mathbf{k}}(\xi) - e^2 \mathcal{A}_{\mathbf{k}}(\xi)^2 - \frac{e}{2}\sigma^{\mu\nu} f_{\mu\nu} \mathcal{A}'_{\mathbf{k}}(\xi).
\end{aligned} \tag{82}$$

In the following, we use latin index notation for the two-dimensional (t, x)-plane. Our goal is to evaluate the matrix element in (80) explicitly solving the quantum mechanical system on the world-line. We have the basic commutators

$$[\pi_a, x_b] = i\eta_{ab}, \qquad [\pi_a, \pi_b] = -ieF_{ab} = 0, \qquad [\pi_a, V] = i\partial_a V. \tag{83}$$

Notice the vanishing of F_{ab} in the (t, x)-plane. This represents a rather important simplification with respect to the direct four-dimensional world-line problem. The two-dimensional world-line equations of motion are:

$$\begin{aligned}
\frac{dx^a}{d\tau} &= i[H, x^a] = -2\pi^a, \\
\frac{d\pi^a}{d\tau} &= i[H, \pi^a] = \partial^a V = n^a V'(\xi).
\end{aligned} \tag{84}$$

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Two important identities that follow are

$$\frac{d}{d\tau}(\pi^a n_a) = 0, \qquad \left[\xi, \frac{d\xi}{d\tau}\right] = 0, \tag{85}$$

so that the quantity

$$-2\pi_a n^a = \frac{\xi(\tau) - \xi(0)}{\tau} \tag{86}$$

is a constant of the motion. Using this, we integrate the equation of motion of π^a:

$$\pi^a = D^a + \frac{\tau}{\xi(\tau) - \xi(0)} n^a V \tag{87}$$

in terms of a constant operator D^a, commuting with $\pi^a n_a$. Using $\pi^a = -\frac{1}{2} dx^a/d\tau$ and integrating further, we get

$$-\frac{1}{2}\left(x^a(\tau) - x^a(0)\right) = D^a \tau + \frac{1}{(2\pi^a n_a)^2} \int_{\xi(0)}^{\xi(\tau)} d\xi \, n^a V(\xi).$$

From here we determine D^a, and plugging it back into the expression for π^a (87):

$$\pi^a = -\frac{x_\tau^a - x_0^a}{2\tau} + \frac{\tau}{\xi_\tau - \xi_0} n^a V(\xi) - \frac{\tau}{(\xi_\tau - \xi_0)^2} \int_{\xi_0}^{\xi_\tau} d\xi \, n^a V(\xi). \tag{88}$$

With these elements we can write H as a function of x_τ^a and x_0^a. In order to evaluate the matrix element (80) we need to order the x_τ^a chronologically (notice that $[\xi_\tau, \xi_0] = 0$ and thus we do not care about their ordering). The basic commutator is

$$[\xi_0, x_\tau^a] = [\xi_\tau + 2(\pi_b n^b)\tau, x^a] = 2n_b \tau [\pi^b, x^a] = 2i n^a \tau.$$

We also need the value of $[x^a(\tau), x_a(0)]$, which can be obtained by solving for x_0^a as a function of x_τ^a:

$$x_0^a = x_\tau^a + 2\tau \, \pi_\tau^a + \frac{\tau}{(\xi_\tau - \xi_0)^2} \int_{\xi_0}^{\xi_\tau} d\xi \, n^a V(\xi).$$

Hence

$$[x_0^a, x_{a\tau}] = 2\tau [\pi^a, x_a] + \left[\frac{\tau}{(\xi_\tau - \xi_0)^2} \int_{\xi_0}^{\xi_\tau} d\xi \, V(\xi), \xi\right] = 2\tau \cdot i\delta_a^a = 4i\tau.$$

Now we can compute the world-line Hamiltonian directly from its definition $H = \pi^a \pi_a + V$ and order it into the form:

$$H = \frac{1}{4\tau^2}\left(x^a(\tau)x_a(\tau) - 2x^a(\tau)x_a(0) + x^a(0)x_a(0)\right) - \frac{i}{\tau} + \frac{1}{\xi_\tau - \xi_0} \int_{\xi_0}^{\xi_\tau} d\xi \, V(\xi). \tag{89}$$

The heat kernel takes the form

$$U_{\mathbf{k}}(t, x; t', x', \tau) = \langle t, x | e^{-i\tau H} | t', x'\rangle = C_{\mathbf{k}}(t, x; t', x') \, e^{-i\int^\tau d\tau' F_{\mathbf{k}}(t,x;t',x',\tau')}, \tag{90}$$

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where

$$F_{\mathbf{k}}(t, x; t', x', \tau) = \frac{1}{4\tau^2}\left[(t - t')^2 - (x - x')^2\right] - \frac{i}{\tau} + \frac{1}{\xi - \xi'}\int_{\xi'}^{\xi} V. \tag{91}$$

We find, upon integrating in τ:

$$U_{\mathbf{k}}(t, x; t', x', \tau) = \frac{C_{\mathbf{k}}(x_a; x'_a)}{\tau} e^{i\left[\frac{(x_a - x'_a)^2}{4\tau} - \frac{\tau}{\xi - \xi'}\int_{\xi'}^{\xi} V\right]}. \tag{92}$$

The τ-independent prefactor is fixed by requiring

$$(i\partial_a - e\mathcal{A}_a)U_{\mathbf{k}}(t, x; t', x', \tau) = (-i\partial'_a - e\mathcal{A}_a)U_{\mathbf{k}}(t, x; t', x', \tau) = \langle t_\tau, x_\tau | \pi_a | t'_0, x'_0 \rangle. \tag{93}$$

These equations imply that $C(x_a, x'_a)$ is covariantly constant in each argument. Hence it is given by a two-dimensioal Wilson line, with the overall scale fixed by requiring

$$\lim_{\tau \to 0} U_{\mathbf{k}}(t, x; t', x', \tau) = \delta(t - t')\,\delta(x - x').$$

Explicitly:

$$C_{\mathbf{k}}(t, x; t', x') = -\frac{1}{4\pi}\,\Phi_{\mathbf{k}}^{(2)}(t, x; t', x') = -\frac{1}{4\pi}\,\exp\left(-ie\int_{(t',x')}^{(t,x)} dz^a \mathcal{A}_a\right). \tag{94}$$

The Wilson line can be further reduced using $\varepsilon^a = \varepsilon^0\, n^a$:

$$\int_{(t',x')}^{(t,x)} dz^a \mathcal{A}_a = \varepsilon^0 \int_{\xi'}^{\xi} d\bar{\xi} \mathcal{A}_{\mathbf{k}}(\bar{\xi}), \tag{95}$$

where we have integrated on a straight line. Evaluating the integral over the effective potential V and putting all factors together, we get the following expression for the four-dimensional bosonic kernel:

$$U(x_\mu; x'_\mu, \tau) = \int \frac{d\mathbf{k}}{(2\pi)^2} U_{\mathbf{k}}(t, x; t', x', \tau)\, e^{i\mathbf{k}(\mathbf{y} - \mathbf{y}')}, \tag{96}$$

where

$$U_{\mathbf{k}}(t, x; t', x', \tau) = -\frac{1}{4\pi\tau}\Phi^{(2)}(t, x; t', x')\, e^{iF^{(2)}} \exp\left(i\tau\mathbf{k}^2 - 2i\tau e\mathbf{k}\cdot\boldsymbol{\varepsilon} \int_{\xi'}^{\xi} \frac{\mathcal{A}_{\mathbf{k}}}{\xi - \xi'}\right),$$

$$F^{(2)}(t, x; t', x', \tau) = \frac{(x_a - x'_a)^2}{4\tau} + \tau m^2 + \tau e^2 \int_{\xi'}^{\xi} \frac{\mathcal{A}_{\mathbf{k}}^2}{\xi - \xi'} + \tfrac{1}{2}\, e\tau f_{\mu\nu}\sigma^{\mu\nu} \frac{\mathcal{A}_{\mathbf{k}}(\xi) - \mathcal{A}_{\mathbf{k}}(\xi')}{\xi - \xi'}. \tag{97}$$

We can check this expression by comparing it with Schwinger's result for $\theta = 0$, where $\mathcal{A}_{\mathbf{k}}(\xi) = A(\xi)$. In this case the $\mathbf{k}$-integral is gaussian:

$$\int \frac{d\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}(\mathbf{y} - \mathbf{y}')}\exp\left(i\tau\mathbf{k}^2 - 2i\tau e\mathbf{k}\cdot\boldsymbol{\varepsilon} \int_{\xi'}^{\xi} A\right)$$

$$= \frac{-i}{4\pi\tau} e^{-i\frac{(\mathbf{y}-\mathbf{y}')^2}{4\tau} - i\tau e^2 \left(\frac{1}{\xi-\xi'} \int_{\xi'}^{\xi} A\right)^2} \exp\left(ie\frac{(\mathbf{y}-\mathbf{y}') \cdot \boldsymbol{\varepsilon}}{\xi - \xi'} \int_{\xi'}^{\xi} A\right). \tag{98}$$

This agrees with Schwinger's result, the last term just giving the rest of the four-dimensional Wilson line, *i.e.* using

$$\int_{x'_\mu}^{x_\mu} dz^\nu = \frac{(x^\nu - x'^\nu)}{\xi - \xi'} \int_{\xi'}^{\xi} d\bar\xi$$

for a straight-line integral, we get

$$-e\int_{x'\to x} A = -e\varepsilon^0 \int_{\xi'}^{\xi} d\bar\xi A(\bar\xi) + e\frac{(\mathbf{y}-\mathbf{y}') \cdot \boldsymbol{\varepsilon}}{\xi - \xi'} \int_{\xi'}^{\xi} d\bar\xi A(\bar\xi),$$

giving the two pieces (95) and (98) of the full Wilson line.

Effective Mass

For a periodic wave $A(\xi) = A \sin(\omega\xi)$ one can define an effective mass describing the effects of the plane wave on the inertial properties of the particle. From the general expression in (97) we read off the effective mass for propagation in the (t, x) plane, as a function of the transverse momentum $\mathbf{k}$, given by

$$m_{\text{eff}}^2(\mathbf{k}) = m^2 + e^2 \lim_{|\xi-\xi'|\to\infty} \int_{\xi'}^{\xi} \frac{\mathcal{A}_{\mathbf{k}}^2}{\xi - \xi'} = m^2 + e^2 \overline{\mathcal{A}_{\mathbf{k}}^2}. \tag{99}$$

In the commutative case $\overline{\mathcal{A}_{\mathbf{k}}^2} = \overline{A^2} = A^2/2$ and the effective mass is actually independent of $\mathbf{k}$. For the non-commutative case, we find exactly the same result for charge-one particles:

$$m_{\text{eff}}^2(\mathbf{k})_{q=1} = m^2 + \tfrac{1}{2} e^2 A^2, \tag{100}$$

whereas the neutral particles show θ-dependent resonant effects:

$$m_{\text{eff}}^2(\mathbf{k})_{q=0} = m^2 + 2e^2 A^2 \sin^2\left(\tfrac{1}{2}\omega n\theta k\right). \tag{101}$$

Namely, the effective mass renormalization can be completely cancelled if a tuning is made of the transverse momentum and the non-commutative deformation parameter.

Effective Action

The one-loop effective action $\delta\mathcal{L}$ is defined by:

$$\exp\left(i \int \delta\mathcal{L}\right) = \frac{\det(i\slashed{D}_{A\star} - m)}{\det(i\slashed{\partial} - m)} = \frac{\det^{\frac{1}{2}}\left(H(\slashed{D}_{A\star})\right)}{\det^{\frac{1}{2}}(-\partial^2 - m^2)}. \tag{102}$$

This quantity was computed by Schwinger in the commutative case and shown to be trivial, in agreement with the expectations from the point of view of effective field theory. Namely all local Lorentz-invariants constructed from the field strength $F_{\mu\nu}$ or its dual $F_{\mu\nu}^*$

vanish in the plane-wave background. In fact, it is not hard to show that all polynomial invariants constructed from powers $\widehat{F}$, $\widehat{F}^*$ and non-commutative covariant derivatives also vanish in the non-commutative plane-wave background. Thus, we expect the effective action (102) to vanish, up to boundary terms.

The proper-time representation is:

$$\delta\mathcal{L}(x) = -\frac{i}{2}\int_{-\infty}^{0}\frac{d\tau}{\tau}\,\text{tr}_{\text{Dirac}}\,U(x;x,\tau) - (\varepsilon = 0), \tag{103}$$

where the $m^2 - i0$ prescription is assumed. We can get rid of it by rotating the contour to euclidean proper-time $\tau = is$, so that

$$\delta\mathcal{L}(x) = \frac{i}{2}\int_{0}^{\infty}\frac{ds}{s}\,\text{tr}_{\text{Dirac}}\,U(x;x,is) - (\varepsilon = 0). \tag{104}$$

Evaluating the kernel (96) in the limit $x \to x'$, $\xi \to \xi'$ we get

$$U(x;x,is) = \frac{i}{4\pi s}e^{-s\left(m^2 + \frac{e}{2}f_{\mu\nu}\sigma^{\mu\nu}\,\mathcal{A}'_{\mathbf{k}}(\xi)\right)}\int\frac{d\mathbf{k}}{(2\pi)^2}\exp\left[-s\left(\mathbf{k} - e\varepsilon\mathcal{A}_{\mathbf{k}}(\xi)\right)^2\right]. \tag{105}$$

Using that, for a plane wave

$$\text{tr}_{\text{Dirac}}e^{-s\frac{e}{2}f\cdot\sigma\mathcal{A}'} = 4, \tag{106}$$

we find the final result for the effective action:

$$\int\delta\mathcal{L}(x) = -\frac{1}{2\pi}\int_{0}^{\infty}\frac{ds}{s^2}e^{-sm^2}\int d^4x\int\frac{d\mathbf{k}}{(2\pi)^2}\exp\left[-s\left(\mathbf{k} - e\varepsilon\mathcal{A}_{\mathbf{k}}(\xi)\right)^2\right] - (\varepsilon = 0). \tag{107}$$

We can analyze the effective action (107) more carefully to show that indeed it will only lead to boundary terms. Surprisingly, the result depends once again on the celebrated IR/UV correspondence of [9]. To define the effective action derived from (107) we need to specify a way to regularize the x- and $\mathbf{k}$-integrals. We will assume the wave-front to be compact supported with a size Δ. Using light-cone variables $\xi = t - x, \eta = t + x$, the volume measure becomes $d\xi\,d\eta\,d\mathbf{y}$. We will consider the space-time integral to be done in a space-time box of volume L^4 (assuming for simplicity $L \gg \Delta$). Since $\mathcal{A}_{\mathbf{k}}(\xi)$ is a pulse centered at $\pm|n\theta k|$ (for $q = 1$) or two pulses centered one at $|n\theta k|$ and the other at $-|n\theta k|$, the momentum integration variable is constrained to satisfy $|n\theta k| \leq L$. Once again the presence of θ translates an infrared cutoff L in x into an ultraviolet cutoff in $\mathbf{k}$.

For each value of $\mathbf{k}$, the integral over ξ can be done, since the argument depends only on ξ. If we are away from the boundary of the box, the value of the integral near $\xi = -\frac{L}{2}$ is

$$C_1 + \xi\,e^{-s\mathbf{k}^2},$$

while near $\xi = \frac{L}{2}$ it is

$$C_2 + \xi\,e^{-s\mathbf{k}^2}$$

and the integral is just the difference between both expressions evaluated at $\pm\frac{L}{2}$:

$$(C_2 - C_1) + L\,e^{-s\mathbf{k}^2}. \tag{108}$$

The value of $C_2 - C_1$ depends on the particular shape of the pulse and on s, but in the limit we are considering, where $\Delta \ll L$, they are slowly varying functions of $\mathbf{k}$. When we perform the integral over $\mathbf{k}$, the first term produces a quadratic divergence $\sim (C_2 - C_1)(L/\theta)^2$ as $L \to \infty$, where θ is the typical eigenvalue of the θ-matrix. This exhibits a mild version of the IR/UV mixing that appears in loop computations in [9, 10].

We see that a careful correlation of ultraviolet and infrared cutoffs, such that the shifted pulses are always 'inside the box', guarantees that (107), being a function of a single space-time variable, is a total derivative. Hence we conclude as in the commutative case that there is no bulk contribution to the effective action.

Acknowledgements

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